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Explicit H_2-Estimates and Pointwise Bounds for Solutions of Second-Order Elliptic Boundary Value Problems

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It is well known that the H_2-norm and the C_0-norm of a function $u \in H_2(\Omega)$ (where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \leqslant 3$) can be estimated in terms of a given uniformly elliptic second-order differential operator L and some boundary operator B applied to u, if certain regularity assumptions are satisfied. If these bounds shall be used for numerical purposes, the constants occurring in the estimates must be known *explicitly*. The main goal of the present article is the computation of such explicit constants. For simplicity of presentation, we restrict ourselves to the case where $L[u] = -\Delta u + c(x)u$. As an application, we prove an *existence and inclusion* result for nonlinear boundary value problems. © 1992 Academic Press, Inc.

1. Introduction

Consider the scalar elliptic boundary value problem

$$-\Delta U(x) + F(x, U(x)) = 0 \;(x \in \Omega), \qquad B[U](x) = s(x) \;(x \in \partial\Omega). \tag{1}$$

Here, $\Omega \subset \mathbb{R}^n$ (with $n \in \{2, 3\}$) is a bounded domain with Lipschitz-continuous boundary $\partial\Omega$ and certain additional properties specified later. F is continuous on $\bar{\Omega} \times \mathbb{R}$, together with its derivative $\partial F/\partial U$. The boundary operator B is of Dirichlet-, Neumann-, or mixed type; $s: \partial\Omega \to \mathbb{R}$ is a function with $s = B[\bar{s}]$ (in the trace sense) for some $\bar{s} \in H_2(\Omega)$.

In the present article, our aim is to establish the main theoretical part of a numerical method which proves the existence of a solution of problem (1) within explicit and "close" bounds, provided that an approximate solution $\omega \in H_2(\Omega)$ can be computed such that $B[\omega] - s$ is essentially bounded on $\partial\Omega$, and

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(i) sufficiently small bounds δ, δ_B for the *defects* of ω can be calculated:

$$\|-\Delta\omega + F(\cdot, \omega)\|_{2,\Omega} \leqslant \delta, \qquad \|B[\omega] - s\|_{\infty,\partial\Omega} \leqslant \delta_B; \tag{2}$$

(ii) constants K and K_B are known *explicitly* which satisfy

$$\|u\|_{\infty,\Omega} \leqslant K\,\|L[u]\|_{2,\Omega} + K_B\,\|B[u]\|_{\infty,\partial\Omega} \qquad \text{for} \quad u \in \hat{H}_2^B(\Omega). \tag{3}$$

Here, $\hat{H}_2^B(\Omega)$ is the space consisting of all $u \in H_2(\Omega)$ such that $B[u]$ is essentially bounded on $\partial\Omega$, and L denotes the linear operator given by

$$L[u] := -\Delta u + c(x)u \qquad (u \in H_2(\Omega)), \tag{4}$$

where $c(x) := (\partial F/\partial U)(x, \omega(x))$ for $x \in \bar{\Omega}$.

If, for example, problem (1) is linear, i.e., $F(x, y) = c(x)\,y - r(x)$ with given continuous functions c and r on $\bar{\Omega}$ and, moreover, $\partial\Omega$ is sufficiently smooth and B is not of mixed type, then (2), (3), and the well-known theory of elliptic boundary value problems (compare Section 5) immediately provide the existence of a (unique) solution $U \in H_2(\Omega)$ of problem (1) satisfying

$$\|U - \omega\|_{\infty,\Omega} \leqslant K\delta + K_B\delta_B.$$

In the general nonlinear case, (2) and (3) may be used in combination with a theorem of Newton–Kantorovich-type to derive the desired existence and inclusion statement. Details will be presented in the final section.

In the remaining sections, we will be concerned with the explicit computation of constants K, K_B satisfying (3). (Of course, (3) or the estimates (5)–(7) stated below may also be of interest in other contexts.) First we use an explicit version of the Sobolev embedding $H_2(\Omega) \hookrightarrow C_0(\bar{\Omega})$ (note that $n \in \{2, 3\}$) to calculate constants C_0, C_1, C_2 such that

$$\|u\|_{\infty,\Omega} \leqslant C_0\,\|u\|_{2,\Omega} + C_1\,\|u_x\|_{2,\Omega} + C_2\,\|u_{xx}\|_{2,\Omega} \qquad \text{for} \quad u \in H_2(\Omega), \tag{5}$$

where $u_x := (\operatorname{grad} u)'$, u_{xx} denotes the Hesse matrix of u, and accordingly, $\|u_x\|_{2,\Omega}^2 = \sum_{i=1}^n \|\partial u/\partial x_i\|_{2,\Omega}^2$, $\|u_{xx}\|_{2,\Omega}^2 = \sum_{i,j=1}^n \|\partial^2 u/\partial x_i\,\partial x_j\|_{2,\Omega}^2$.

In a second step, we compute constants K_0, K_1, K_2 satisfying

$$\begin{aligned} &\|u\|_{2,\Omega} \leqslant K_0\,\|L[u]\|_{2,\Omega}, \qquad \|u_x\|_{2,\Omega} \leqslant K_1\,\|L[u]\|_{2,\Omega}, \\ &\|u_{xx}\|_{2,\Omega} \leqslant K_2\,\|L[u]\|_{2,\Omega} \end{aligned} \tag{6}$$

for $u \in H_2^B(\Omega)$. The latter space is defined to be the closure in $H_2(\Omega)$ of all functions $u \in C_2(\bar{\Omega})$ satisfying $B[u] = 0$ almost everywhere on $\partial\Omega$.

Of course, the estimates (6) require L to be *invertible* on $H_2^B(\Omega)$. To compute K_0 we use *eigenvalue estimates* which have, in general, to be carried out numerically. K_1 and K_2 may be calculated in a more direct way.

Combining (5) and (6) we find that

$$\|u\|_{\infty, \Omega} \leqslant K \, \|L[u]\|_{2, \Omega} \qquad \text{for} \quad u \in H_2^B(\Omega), \tag{7}$$

where $K := C_0 K_0 + C_1 K_1 + C_2 K_2$. Finally, we use the technique of weak differential inequalities (for some shifted operator $L + \mu I$) to obtain (3) in full generality.

We wish to remark that, without fundamental difficulties but with more technical effort (and with less clearness of the resulting formulas), the methods of computing the constants in (6), (7), and (3) may be carried over to more general uniformly elliptic operators of the form

$$L[u] = -\sum_{i, j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u$$

and to more general boundary operators B.

The existence and inclusion method described in the final section has been applied successfully to several examples with *ordinary* differential equations in [4, 17, 18], and, with (3) (or (7)) replaced by a different method of estimating the inverse operator, in [9]. Numerical examples of nonlinear *elliptic* boundary value problems are presented in [20] and further forthcoming papers.

It should be noted that the method does not require any kinds of monotonicity assumptions (or growth conditions) on the nonlinearity F, or inverse-positivity assumptions on the operator L. Thus, important cases are covered where the classical method of upper and lower solutions (see [21] for a survey of literature) cannot be applied.

2. Explicit Embedding Constants

Since $n \in \{2, 3\}$ and $\partial\Omega$ is Lipschitz-continuous, the well-known Sobolev embedding theorem [1, Theorem 5.4, pp. 97, 98] provides the continuous embedding $H_2(\Omega) \hookrightarrow C_0(\bar{\Omega})$. Here, we will prove an *explicit* version of that theorem; i.e., we compute constants C_0, C_1, C_2 satisfying (5). Essentially, our proof follows the lines of the "theoretical" proof. However, it turns out to be advantageous to replace the *spherical cones* used there by more general convex sets Q, in order to obtain smaller constants.

THEOREM 1. *For fixed $x_0 \in \bar{\Omega}$, let $Q \subset \bar{\Omega}$ denote a closed convex set such* *that* $\operatorname{int}(Q) \neq \emptyset$ *and* $x_0 \in Q$. *Moreover, let the (modified) momenta of* Q *with respect to* x_0 *be defined by*

$$M_\nu(Q, x_0) := \left[\frac{1}{\operatorname{meas}(Q)} \int_Q |x - x_0|^{2\nu}\, dx\right]^{1/2} \qquad (\nu \in \{1, 2\}). \tag{8}$$

Then the following assertions hold true:

(a) *For* $u \in H_2(\Omega)$ *(more precisely, for the continuous representative of* u*),*

$$|u(x_0)| \leqslant \frac{1}{\sqrt{\operatorname{meas}(Q)}} \cdot [\gamma_0 \, \|u\|_{2,\Omega} + \gamma_1 M_1(Q, x_0) \, \|u_x\|_{2,\Omega} + \gamma_2 M_2(Q, x_0) \, \|u_{xx}\|_{2,\Omega}], \tag{9}$$

where

$$\begin{array}{llll} \gamma_0 = 1, & \gamma_1 = 1.1548, & \gamma_2 = 0.22361 & \textit{if} \quad n = 2, \\ \gamma_0 = 1.0708, & \gamma_1 = 1.6549, & \gamma_2 = 0.41413 & \textit{if} \quad n = 3. \end{array} \tag{10}$$

(b) *If* Ω *is convex and* $Q = \bar{\Omega}$, *then* (9) *holds for* $u \in H_2^{B_0}(\Omega)$ *(where* B_0 *is the Dirichlet boundary operator) with*

$$\begin{array}{llll} \gamma_0 = 0, & \gamma_1 = 1.4143, & \gamma_2 = 0.35356 & \textit{if} \quad n = 2, \\ \gamma_0 = 0, & \gamma_1 = 1.4908, & \gamma_2 = 0.50918 & \textit{if} \quad n = 3. \end{array} \tag{11}$$

Proof. Because the embedding $H_2(\Omega) \subsetneq C_0(\bar{\Omega})$ is continuous and $C_2(\bar{\Omega})$ is dense in $H_2(\Omega)$, it suffices to prove the assertion for $u \in C_2(\bar{\Omega})$. To prove part (b) we may, in addition, assume that $u(x) = 0$ for $x \in \partial\Omega$. Since $Q = \overline{\operatorname{int}(Q)}$ and both sides of inequality (9) depend continuously on x_0, we may assume that $x_0 \in \operatorname{int}(Q)$.

Let S_n denote the unit sphere in $\mathbb{R}^n$ (with respect to the Euclidean norm $|\cdot|$). Due to the properties of Q, there exists a continuous mapping $R: S_n \to (0, \infty)$ such that, for each $\omega \in S_n$,

$$x_0 + r\omega \begin{cases} \in \operatorname{int}(Q) & \text{for} \quad 0 \leqslant r < R(\omega) \\ \in \partial Q & \text{for} \quad r = R(\omega) \\ \notin Q & \text{for} \quad r > R(\omega). \end{cases}$$

To prove part (a) let some real $\alpha > \frac{3}{2}$ be chosen; to prove part (b) let $\alpha := 1$. Then, for each $\omega \in S_n$,

$$\begin{aligned}R(\omega)^n u(x_0) &= -R(\omega)^{n-\alpha} \int_0^{R(\omega)} \frac{\partial}{\partial r} \{(R(\omega)-r)^\alpha u(x_0+r\omega)\}\, dr \\ &= -R(\omega)^{n-\alpha} \left[r \frac{\partial}{\partial r} \{(R(\omega)-r)^\alpha u(x_0+r\omega)\} \right]_0^{R(\omega)} \\ &\quad + R(\omega)^{n-\alpha} \int_0^{R(\omega)} r \frac{\partial^2}{\partial r^2} \{(R(\omega)-r)^\alpha u(x_0+r\omega)\}\, dr.\end{aligned}$$

Here, the boundary terms vanish due to the choice of α. Integrating over $\omega \in S_n$ we therefore obtain, using the transformation $r=|x-x_0|$, $\omega=(x-x_0)/r$, $r^{n-1}\, dr\, d\omega = dx$:

$$\begin{aligned}&\left(\int_{S_n} R(\omega)^n\, d\omega\right) \cdot u(x_0) \\ &\quad = \int_Q R(\omega)^{n-\alpha} r^{2-n} \frac{\partial^2}{\partial r^2} \{(R(\omega)-r)^\alpha u(x_0+r\omega)\}\, dx \\ &\quad = \alpha(\alpha-1) \int_Q R(\omega)^{n-\alpha} r^{2-n} (R(\omega)-r)^{\alpha-2} u(x)\, dx \\ &\qquad -2\alpha \int_Q R(\omega)^{n-\alpha} r^{2-n} (R(\omega)-r)^{\alpha-1} [\omega^t \cdot u_x(x)]\, dx \\ &\qquad + \int_Q R(\omega)^{n-\alpha} r^{2-n} (R(\omega)-r)^{\alpha} [\omega^t \cdot u_{xx}(x) \cdot \omega]\, dx.\end{aligned}$$

Here, the first of the three terms on the right-hand side does not occur if $\alpha=1$. Using the estimates $|\omega^t \cdot u_x(x)| \leqslant [\sum_{i=1}^n (\partial u/\partial x_i)(x)^2]^{1/2}$, $|\omega^t \cdot u_{xx}(x) \cdot \omega| \leqslant [\sum_{i,j=1}^n (\partial^2 u/\partial x_i\, \partial x_j)(x)^2]^{1/2}$, and Schwarz's inequality, we obtain

$$\begin{aligned}&\left(\int_{S_n} R(\omega)^n\, d\omega\right) |u(x_0)| \\ &\quad \leqslant \alpha(\alpha-1) \left[\int_Q R(\omega)^{2n-2\alpha} r^{4-2n} (R(\omega)-r)^{2\alpha-4}\, dx\right]^{1/2} \cdot \|u\|_{2,Q} \\ &\qquad + 2\alpha \left[\int_Q R(\omega)^{2n-2\alpha} r^{4-2n} (R(\omega)-r)^{2\alpha-2}\, dx\right]^{1/2} \cdot \|u_x\|_{2,Q} \\ &\qquad + \left[\int_Q R(\omega)^{2n-2\alpha} r^{4-2n} (R(\omega)-r)^{2\alpha}\, dx\right]^{1/2} \cdot \|u_{xx}\|_{2,Q}. \qquad (12)\end{aligned}$$

Transforming back to polar coordinates we derive in a straightforward way, for $\nu \in \{0, 2, 4\}$ and $\alpha > (\nu-1)/2$,

$$\int_Q R(\omega)^{2n-2\alpha} r^{4-2n}(R(\omega)-r)^{2\alpha-\nu}\, dx = \beta_n(\alpha, \nu) \cdot \int_Q r^{4-\nu}\, dx, \tag{13}$$

where $\beta_2(\alpha, \nu) := (6-\nu)/[(2\alpha-\nu+1)(2\alpha - \nu + 2)]$, $\beta_3(\alpha, \nu) := (7 - \nu)/(2\alpha-\nu+1)$. Inserting (13) into (12) and using the inclusion $Q \subset \bar{\Omega}$ and the identity $\int_{S_n} R(\omega)^n\, d\omega = n \cdot \mathrm{meas}(Q)$, we obtain the estimate (9) where, in the case $\alpha > \frac{3}{2}$,

$$\begin{aligned}
&\gamma_0 = \frac{1}{2}\,\alpha \sqrt{\frac{\alpha-1}{2\alpha-3}}, \quad \gamma_1 = \sqrt{\frac{2\alpha}{2\alpha-1}}, \quad \gamma_2 = \frac{1}{2}\sqrt{\frac{3}{(2\alpha+1)(\alpha+1)}} \quad \text{if } n=2,\\
&\gamma_0 = \frac{\alpha(\alpha-1)}{\sqrt{3\cdot(2\alpha-3)}}, \quad \gamma_1 = \frac{2}{3}\,\alpha\sqrt{\frac{5}{2\alpha-1}}, \quad \gamma_2 = \frac{1}{3}\sqrt{\frac{7}{2\alpha+1}} \quad \text{if } n=3;
\end{aligned} \tag{14}$$

in the alternative case where $\alpha = 1$, (9) holds true with γ_1 and γ_2 defined in (14), but with $\gamma_0 = 0$ (note that the first of the three terms on the right-hand side of (12) does not occur in that case).

Inserting $\alpha = 1$ into γ_1 and γ_2 in (14) we obtain the values given in (11) as upper bounds. Part (b) of the theorem is therefore proved. To prove part (a) we choose $\alpha > \frac{3}{2}$, minimizing the term γ_0 in (14); i.e., $\alpha = 2$ if $n = 2$ and $\alpha = (7+\sqrt{13})/6$ if $n = 3$. Inserting these numbers into (14) we obtain the values given in (10) as upper bounds. ∎

Remarks. (1) The proof of Theorem 1 shows that the values γ_0, γ_1, γ_2 in (10) may be replaced by (14) with arbitrary $\alpha > \frac{3}{2}$. In particular, γ_2 may be made arbitrarily small if γ_0 (and, for $n = 3$, also γ_1) are allowed to become "large."

(2) Theorem 1 is formulated for fixed $x_0 \in \bar{\Omega}$. However, it is not difficult to derive the uniform estimate (5) from this theorem: Because $\partial\Omega$ is Lipschitz-continuous, one can choose a family $(Q(x_0))_{x_0 \in \bar{\Omega}}$ of closed convex sets $Q(x_0)$ such that $x_0 \in Q(x_0) \subset \bar{\Omega}$ for each $x_0 \in \bar{\Omega}$ and, moreover, $\mathrm{meas}(Q(x_0)) \geqslant q_0 > 0$ for $x_0 \in \bar{\Omega}$. For example, $Q(x_0)$ may be chosen to be a *spherical cone* with vertex at x_0, as in the "theoretical" proof of the embedding theorem or in [13], where explicit constants have been computed for this particular choice. Since the momenta $M_\nu(Q(x_0), x_0)$ are bounded from above (for instance, by $[\mathrm{diam}(\Omega)]^\nu$), (5) follows. The following corollary deals with the particular choice where all the sets $Q(x_0)$ are *congruent images* of one fixed set Q.

Corollary 1. *Let $Q \subseteq \mathbb{R}^n$ denote a compact convex set with* $\operatorname{int}(Q) \neq \varnothing$ *and define, with $M_\nu(Q, x_0)$ given by* (8),

$$\bar{M}_\nu(Q) := \max_{x_0 \in Q} M_\nu(Q, x_0) \qquad (\nu \in \{1, 2\}).$$

Suppose that, for each $x_0 \in \bar{\Omega}$, a congruent image of Q has the properties required in Theorem 1, *i.e., there exists an orthogonal matrix $T \in \mathbb{R}^{n,n}$ and some $b \in \mathbb{R}^n$ (both possibly depending on x_0) such that*

$$x_0 \in \varphi(Q) \subset \bar{\Omega} \qquad \textit{for} \quad \varphi(x) := Tx + b \ (x \in \mathbb{R}^n). \tag{15}$$

Then, for each $u \in H_2(\Omega)$,

$$\|u\|_{\infty, \Omega} \leqslant \frac{1}{\sqrt{\operatorname{meas}(Q)}} \left[\gamma_0 \|u\|_{2, \Omega} + \gamma_1 \bar{M}_1(Q) \|u_x\|_{2, \Omega} + \gamma_2 \bar{M}_2(Q) \|u_{xx}\|_{2, \Omega}\right]$$

with γ_0, γ_1, γ_2 given by (10).

Proof. The assertion follows immediately from Theorem 1 and the fact that, for $x_0 \in \bar{\Omega}$ and φ satisfying (15), $\operatorname{meas}(\varphi(Q)) = \operatorname{meas}(Q)$ and

$$M_\nu(\varphi(Q), x_0) = M_\nu(Q, \varphi^{-1}(x_0)) \leqslant \bar{M}_\nu(Q). \quad \blacksquare$$

Examples. (1) Let Ω be an (open) ball of radius R. Then, Corollary 1 can be applied with Q denoting a (closed) ball of radius $\rho \in (0, R]$. Straightforward calculations show that

$$\bar{M}_1(Q) = \sqrt{\frac{3}{2}}\,\rho, \qquad \bar{M}_2(Q) = \sqrt{\frac{10}{3}}\,\rho^2 \qquad \text{if} \quad n = 2;$$

$$\bar{M}_1(Q) = \sqrt{\frac{8}{5}}\,\rho, \qquad \bar{M}_2(Q) = \sqrt{\frac{24}{7}}\,\rho^2 \qquad \text{if} \quad n = 3.$$

According to Corollary 1, the estimate (5) therefore holds, for any $\rho \in (0, R]$, with

$$C_0 = 0.56419 \cdot \rho^{-1}, \quad C_1 = 0.79789, \quad C_2 = 0.23033 \cdot \rho \quad \text{if } n = 2,$$

$$C_0 = 0.52319 \cdot \rho^{-3/2}, \quad C_1 = 1.0228 \cdot \rho^{-1/2}, \quad C_2 = 0.37467 \cdot \rho^{1/2} \quad \text{if } n = 3.$$

(2) Let Ω be a rectangle with sidelengths $L_1, \ldots, L_n$ and choose Q to be a rectangle with sidelengths $l_i \in (0, L_i]$ ($i = 1, \ldots, n$). Simple calculations

show that $\bar{M}_1(Q)=[\frac{1}{3}\sum_{i=1}^n l_i^2]^{1/2}$, $\bar{M}_2(Q)=\frac{1}{3}[(\sum_{i=1}^n l_i^2)^2+\frac{4}{5}\sum_{i=1}^n l_i^4]^{1/2}$ so that Corollary 1 provides the estimate (5) with

$$\begin{gathered}C_0=\gamma_0\cdot\frac{1}{\sqrt{l_1\cdot l_2(\cdot l_3)}},\qquad C_1=\frac{\gamma_1}{\sqrt{3}}\cdot\sqrt{\frac{l_1^2+l_2^2(+l_3^2)}{l_1\cdot l_2(\cdot l_3)}},\\ C_2=\frac{\gamma_2}{3}\cdot\sqrt{\frac{[l_1^2+l_2^2(+l_3^2)]^2+\frac{4}{5}[l_1^4+l_2^4(+l_3^4)]}{l_1\cdot l_2(\cdot l_3)}},\end{gathered}\tag{16}$$

with γ_0, γ_1, γ_2 given by (10).

(3) For the L-shaped domain $\Omega:=(-1,1)^2\backslash[0,1)^2$, Corollary 1 applies for any rectangle Q with sidelengths $l_1\in(0,2]$, $l_2\in(0,1]$. The estimate (5) therefore holds with C_0, C_1, C_2 given by (16).

3. H_1-ESTIMATES

For the following four sections, let L denote a linear operator of the form (4), with a given function $c\in L_\infty(\Omega)$. Moreover, let Γ_0 denote a closed subset of $\partial\Omega$, and $\Gamma_1:=\partial\Omega\backslash\Gamma_0$. Suppose that the boundary operator B is given by

$$B[u]:=\begin{cases}u & \text{on } \Gamma_0\\ \dfrac{\partial u}{\partial \nu} & \text{on } \Gamma_1\end{cases}\qquad \text{for}\quad u\in H_2(\Omega)\tag{17}$$

with $\partial u/\partial\nu:=(\operatorname{grad} u)\cdot\nu\in L_2(\partial\Omega)$, where $\nu\in L_\infty^n(\partial\Omega)$ denotes the outer unit normal at $\partial\Omega$ (which exists almost everywhere on $\partial\Omega$ due to the Lipschitz-continuity of $\partial\Omega$; see [14, Lemma 4.2, p. 88]).

It should be noted that the assumption $n\leqslant 3$ may be dropped for the present and the following section. Moreover, the results in the present section hold true with $H_2^B(\Omega)$ (defined in Section 1) replaced by the (possibly larger) space consisting of all functions $u\in H_2(\Omega)$ such that $B[u]=0$ on $\partial\Omega$ *in the trace sense.*

In this section, we will be concerned with the first two inequalities in (6). According to eigenvalue theory, the estimate

$$\|u\|_{2,\Omega}\leqslant K_0\,\|L[u]\|_{2,\Omega}\qquad (u\in H_2^B(\Omega))\tag{18}$$

holds, if L is invertible on $H_2^B(\Omega)$, for each $K_0>0$ satisfying

$$\frac{1}{K_0}\leqslant|\lambda|\qquad \text{for each eigenvalue } \lambda \text{ of } L \text{ on } H_2^B(\Omega).\tag{19}$$

Here, in the general case (where, for example, Ω may have reentrant corners), the *weak* formulation of the eigenvalue problem must be considered.

The calculation of K_0 via (19) requires the estimation of the eigenvalues of L (on $H_2^B(\Omega)$) neighboring 0.

If, for instance, the eigenvalues μ_j ($j \in \mathbb{N}$) of $-\Delta$ on $H_2^B(\Omega)$ (ordered by magnitude) are known explicitly, and constants $\underline{c}$, $\bar{c}$ exist which satisfy

$$\underline{c} \leqslant c(x) \leqslant \bar{c} \qquad (x \in \bar{\Omega}) \tag{20}$$

and

$$-\mu_j < \underline{c} \leqslant \bar{c} < -\mu_{j-1} \qquad \text{for some} \quad j \in \mathbb{N} \tag{21}$$

(where $\mu_0 := -\infty$), then a corollary of Courant's maximum-minimum principle [6, Theorem 7, p.411] implies $\lambda_{j-1} \leqslant \mu_{j-1} + \bar{c} < 0 < \mu_j + \underline{c} \leqslant \lambda_j$ for the corresponding eigenvalues λ_{j-1}, λ_j of L. Consequently, (19) (and thus, (18)) holds for

$$K_0 := [\min\{-(\mu_{j-1} + \bar{c}), \mu_j + \underline{c}\}]^{-1}. \tag{22}$$

In particular, $K_0 := (\mu_1 + \underline{c})^{-1}$ may be chosen if $\underline{c} > -\mu_1$.

If (21) does not hold or the eigenvalues μ_j of $-\Delta$ are unknown, or if K_0 given by (22) appears to be too large, one will apply *numerical* eigenvalue estimation techniques to compute K_0 via (19). Here, one may use the method presented in [16, 19], where the given eigenvalue problem is connected, by a numerical (stepwise) homotopy, to a "simple" problem with known eigenvalues, or the method developed by Goerisch and Albrecht (e.g., [10]) which is closely related to the Lehmann–Maehly method, or the method of intermediate problems by Bazley and Fox (e.g., [3]).

The computation of a constant K_1 satisfying

$$\|u_x\|_{2,\Omega} \leqslant K_1 \, \|L[u]\|_{2,\Omega} \qquad (u \in H_2^B(\Omega)) \tag{23}$$

may easily be carried out by use of the following:

THEOREM 2. *Suppose that* (18) *and* (20) *hold. Then,* (23) *is true for*

$$K_1 := \begin{cases} [K_0(1 - \underline{c}K_0)]^{1/2} & \text{if} \quad \underline{c}K_0 \leqslant \frac{1}{2} \\ 1/(2\sqrt{\underline{c}}) & \text{otherwise.} \end{cases}$$

Proof. It suffices to prove (23) for $u \in C_2(\bar{\Omega}) \setminus \{0\}$ satisfying $B[u] = 0$ almost everywhere on $\partial\Omega$. For such u, Green's formula (which holds due to the Lipschitz-continuity of $\partial\Omega$; see [14, Theorem 1.1, p. 121]) and the boundary condition for u imply

$$\int_\Omega u(x)\, L[u](x)\, dx = \int_\Omega u_x(x)^t\, u_x(x)\, dx + \int_\Omega c(x)\, u(x)^2\, dx$$

$$\geqslant \|u_x\|^2_{2,\Omega} + \underline{c}\, \|u\|^2_{2,\Omega}.$$

Applying here Schwarz's inequality on the left-hand side we obtain

$$\|u_x\|^2_{2,\Omega} \leqslant \|u\|_{2,\Omega} \cdot \|L[u]\|_{2,\Omega} - \underline{c}\, \|u\|^2_{2,\Omega} = \mu(1-\underline{c}\mu) \cdot \|L[u]\|^2_{2,\Omega}, \tag{24}$$

where $\mu := \|u\|_{2,\Omega}/\|L[u]\|_{2,\Omega}$. (18) shows that $\mu \leqslant K_0$. Calculation of the maximum of the quadratic expression in μ in (24) on $[0, K_0]$ yields the asserted estimate. ∎

4. H_2-Estimates

In this section, we will show how a constant K_2 satisfying

$$\|u_{xx}\|_{2,\Omega} \leqslant K_2\, \|L[u]\|_{2,\Omega} \qquad (u \in H_2^B(\Omega)) \tag{25}$$

can be computed. Together with the results of the preceding section, the H_2-estimates (6) are complete. We will now assume that

$$\Gamma_0 \text{ and } \Gamma_1 \text{ are piecewise } C_2\text{-hypersurfaces} \tag{26}$$

with Γ_0, Γ_1 defining the boundary operator B via (17). Assumption (26) means precisely that a measure-zero subset $Z \subset \partial\Omega$ exists such that $\Gamma_0 \backslash Z$ and $\Gamma_1 \backslash Z$ are open subsets of $\partial\Omega$ and C_2-hypersurfaces of $\mathbb{R}^n$. (Of course, one of them may be empty.) Consequently, the following differential geometrical terms are defined for almost all $x \in \partial\Omega$: the tangential space T_x, the directional derivative $(\partial\nu/\partial v)(x)$ of the outer unit normal field ν in the direction of $v \in T_x$, the second fundamental tensor $S_x\colon T_x \to T_x$ given by $S_x v := -(\partial\nu/\partial v)(x)$ for $v \in T_x$,

the *mean curvature* $H(x) := \operatorname{trace}(S_x)/(n-1)$,

the *normal curvature* $N(x, v) := v^t S_x v / v^t v$ in the direction of $v \in T_x \backslash \{0\}$,

the *maximal principal curvature* $P(x) := \max\{N(x, v) : v \in T_x \backslash \{0\}\}$.

In particular, $H(x) = P(x) = [\nu(x)^t\, \psi''(0)]/[\psi'(0)^t\, \psi'(0)]$ if $n = 2$, with $\psi\colon (-\varepsilon, \varepsilon) \to \partial\Omega$ denoting a local parametrization of $\partial\Omega$ satisfying $\psi(0) = x$.

We suppose that a Lipschitz-continuous function $f\colon \bar{\Omega} \to \mathbb{R}^n$ exists (which therefore has weak derivatives $\partial f_i/\partial x_j \in L_\infty(\Omega)$) such that

$$\begin{aligned} f(x)^t\, \nu(x) &\geqslant (n-1)\, H(x) \qquad \text{for almost all} \quad x \in \Gamma_0, \\ -f(x)^t\, \nu(x) &\geqslant P(x) \qquad \text{for almost all} \quad x \in \Gamma_1, \end{aligned} \tag{27}$$

and that nonnegative constants F_0, F_1 are known which satisfy

$$[f(x)^t f(x)]^{1/2} \leqslant F_0, \tag{28}$$

$$-\operatorname{div} f(x) + \lambda_{\max}[D[f](x) + D[f](x)^t] \leqslant F_1 \qquad \text{for almost all} \quad x \in \Omega,$$

where $D[f](x)$ denotes the Jacobian matrix of f at x and $\lambda_{\max}[M]$ the maximal eigenvalue of the symmetric matrix $M \in \mathbb{R}^{n,n}$.

THEOREM 3. *Suppose that* (20), (18), (23), (26), (27), *and* (28) *hold with constants* $\underline{c}$, $\bar{c}$, K_0, K_1, F_0, F_1. *Then,* (25) *is true for*

$$K_2 := \sqrt{\kappa^2 + 2\kappa F_0 K_1 + F_1 K_1^2}, \qquad \textit{where} \quad \kappa := 1 + K_0 \cdot \max\{\tfrac{1}{2}(\bar{c} - \underline{c}), -\underline{c}\}.$$

Before proving the theorem we will formulate an important corollary providing a particularly simple result for a large class of domains.

COROLLARY 2. *Suppose that* (20) *and* (18) *hold with constants* $\underline{c}$, $\bar{c}$, K_0 *and, moreover, that* Ω *is convex and satisfies* (26), *or* $\partial\Omega$ *consists of* (*one or several*) *polyhedra* (*and is Lipschitz-continuous*). *Then,* (25) *holds for*

$$K_2 := 1 + K_0 \cdot \max\{\tfrac{1}{2}(\bar{c} - \underline{c}), -\underline{c}\}.$$

Proof. For the two stated types of domains, the curvatures $H(x)$ and $P(x)$ are *nonpositive* for almost all $x \in \partial\Omega$. Consequently, (27) and (28) hold for $f \equiv 0$, $F_0 = F_1 = 0$, so that Theorem 3 provides the assertion of the corollary. ∎

Remarks. (1) Obviously, the class of domains with nonpositive boundary-curvatures H and P is even larger than the class considered in Corollary 2. For example, $H \leqslant 0$ and $P \leqslant 0$ for the non-convex circular sector $\Omega := \{(r \cos\varphi, r \sin\varphi) : 0 < r < R,\ 0 < \varphi < \tfrac{3}{2}\pi\}$ or for the union Ω of two circular discs.

(2) In the general case where (non-zero) subsets of $\partial\Omega$ with positive curvature occur, (27) is often satisfied for a function of the form

$$f(x) = \sum_{i=1}^{M} \alpha_i(x)(x - x^{(i)}) \qquad (x \in \bar{\Omega}) \tag{29}$$

with scalar functions (or constants) α_i and fixed points $x^{(i)} \in \mathbb{R}^n$ placed suitably.

If, for example, Ω is "strictly" *star-shaped* with respect to some $x^{(1)} \in \mathbb{R}^n$ in the sense that $\mu(x) := \nu(x)^t\,(x - x^{(1)}) \geqslant \mu_0 > 0$ for almost all $x \in \partial\Omega$ and, moreover, H and P are essentially bounded from above on Γ_0 and Γ_1, respectively, then (27) is satisfied for f given by (29) with $M = 1$ and

$$\alpha_1 := \operatorname*{ess\,sup}_{x \in \partial\Omega} \left\{\frac{(n-1)\,H(x)}{\mu(x)}\right\} \qquad \text{if} \quad \partial\Omega = \Gamma_0,$$

$$\alpha_1 := -\operatorname*{ess\,sup}_{x \in \partial\Omega} \left\{\frac{P(x)}{\mu(x)}\right\} \qquad \text{if} \quad \partial\Omega = \Gamma_1.$$

For domains Ω with several "local centers," one will use (29) with $M \geqslant 2$ and (possibly non-constant) functions α_i.

(3) Condition (27) excludes certain cases of mixed boundary conditions: If some $x^* \in \Gamma_0 \cap \bar{\Gamma}_1$ belonging to a C_2-smooth part of $\partial\Omega$ exists such that H and P are *positive* at x^*, then (27) cannot be satisfied.

Points $x^* \in \Gamma_0 \cap \bar{\Gamma}_1$ forming a *corner* of $\partial\Omega$, however, are (in general) not forbidden by (27), even if H and P are positive in respective neighborhoods of x^*.

(4) If $\Gamma_0 \cap \bar{\Gamma}_1 = \emptyset$ and the curvatures H and P are essentially bounded from above on Γ_0 and Γ_1, respectively, then a function f satisfying (27) always *exists*, as can be derived from [11, Lemma 1.5.1.9, p. 40].

Proof of Theorem 3. The proof will be divided into four lemmata. The main ideas contained in the first two of them have already been used by Grisvard [11] and, in the case $\partial\Omega = \Gamma_0$, by Ladyzhenskaya [12].

LEMMA 1. $\|u_{xx}\|^2_{2,\Omega} = \|\Delta u\|^2_{2,\Omega} + \int_{\partial\Omega} R[u]\,d\sigma$ *for* $u \in C_2(\bar{\Omega})$, *where*

$$R[u] := -(\Delta u)\,\frac{\partial u}{\partial \nu} + \nu^t \cdot u_{xx} \cdot u_x \qquad \text{on } \partial\Omega. \tag{30}$$

Proof. First let $u \in C_3(\bar{\Omega})$. Integrating by parts twice we obtain

$$\begin{aligned}
\|\Delta u\|^2_{2,\Omega} &= \int_\Omega (\Delta u) \cdot (\operatorname{div} u_x)\,dx = \int_{\partial\Omega} (\Delta u)\,\frac{\partial u}{\partial \nu}\,d\sigma - \int_\Omega (\Delta u)^t_x\,u_x\,dx \\
&= \int_{\partial\Omega} (\Delta u)\,\frac{\partial u}{\partial \nu}\,d\sigma - \sum_{i=1}^n \int_\Omega \Delta\left(\frac{\partial u}{\partial x_i}\right) \cdot \frac{\partial u}{\partial x_i}\,dx \\
&= -\int_{\partial\Omega} R[u]\,d\sigma + \sum_{i=1}^n \int_\Omega \left(\frac{\partial u}{\partial x_i}\right)^t_x \left(\frac{\partial u}{\partial x_i}\right)_x dx \\
&= -\int_{\partial\Omega} R[u]\,d\sigma + \|u_{xx}\|^2_{2,\Omega}.
\end{aligned}$$

Thus, the assertion holds for $u \in C_3(\bar{\Omega})$. Now let some $u \in C_2(\bar{\Omega})$ be given. Using a slight modification of Calderón's extension theorem (see [2, Theorem 11.12., p. 171, and the remarks after the proof]) we extend u, as a C_2-function, to some neighborhood of $\bar{\Omega}$. Now the well-known mollifier-technique may be applied to construct a sequence (u_m) of functions $u_m \in C_\infty(\bar{\Omega})$ which converges to u in the Banach-space-norm of $C_2(\bar{\Omega})$. Since the asserted equality holds for each u_m and involves derivatives up to the second order only, it holds for u. ∎

LEMMA 2. *Let $u \in C_2(\bar{\Omega})$ satisfy $B[u] = 0$ almost everywhere on $\partial\Omega$ and let $R[u]$ be given by* (30). *Then,*

$$R[u] = \begin{cases} (n-1)\, H \cdot (\partial u/\partial \nu)^2 & \text{almost everywhere on } \Gamma_0 \\ N(\cdot, u_x) \cdot u_x^t u_x \ (:= 0 \text{ if } u_x = 0) & \text{almost everywhere on } \Gamma_1. \end{cases}$$

Proof. We prove the asserted equality for each fixed $x \in \partial\Omega \backslash Z$, with Z denoting the measure-zero-set introduced after (26). For such x, we can find a neighborhood $V \subset \mathbb{R}^n$ such that $V \cap \partial\Omega \subset \Gamma_0 \backslash Z$ or $V \cap \partial\Omega \subset \Gamma_1 \backslash Z$, and moreover, a neighborhood $U \subset \mathbb{R}^{n-1}$ of $0 \in \mathbb{R}^{n-1}$ and a local C_2-parametrization $\psi: U \to V \cap \partial\Omega$ of $\partial\Omega \backslash Z$ satisfying $\psi(0) = x$. We may choose ψ in such a way that the columns of the Jacobian matrix $D[\psi](0)$ form an *orthonormal* base of T_x. First we prove that

$$\begin{aligned} S_x &= -D[\nu \circ \psi](0) \cdot D[\psi](0)^t \\ &= D[\psi](0) \cdot \left(\sum_{k=1}^{n} \nu_k(x)\, D^2[\psi_k](0) \right) \cdot D[\psi](0)^t \end{aligned} \tag{31}$$

for the second fundamental tensor S_x, with $D^2[\psi_k]$ denoting the Hesse matrix of ψ_k, the kth component of ψ.

Let $v \in T_x$ be given. Consequently, $v = D[\psi](0) w$ for some $w \in \mathbb{R}^{n-1}$. Multiplying this equation by $D[\psi](0)^t$ and using the orthonormality of the columns of $D[\psi](0)$ we obtain $w = D[\psi](0)^t v$. Thus, by definition, $S_x v = -(\partial \nu / \partial v)(x) = -D[\nu \circ \psi](0) w = -D[\nu \circ \psi](0) \cdot D[\psi](0)^t v$ which proves the first equality in (31). To show the second we first observe that $(\nu \circ \psi)^t \cdot (\nu \circ \psi) \equiv 1$ and $D[\psi]^t \cdot (\nu \circ \psi) \equiv 0$ on U. Differentiating these identities and then evaluating at 0 we obtain

$$\begin{aligned} \nu(x)^t \cdot D[\nu \circ \psi](0) &= 0, \\ D[\psi](0)^t \cdot D[\nu \circ \psi](0) &= -\sum_{k=1}^{n} \nu_k(x)\, D^2[\psi_k](0). \end{aligned} \tag{32}$$

The first of these equations shows that the columns of $D[\nu\circ\psi](0)$ belong to T_x and thus, for some $(n-1)\times(n-1)$-matrix W,

$$D[\nu\circ\psi](0)=D[\psi](0)\cdot W. \tag{33}$$

Inserting (33) into the second equation in (32) and using that $D[\psi](0)' \cdot D[\psi](0)$ is the identity matrix we obtain $W = -\sum_{k=1}^{n} \nu_k(x)\, D^2[\psi_k](0)$. Thus, (33) implies the second equality in (31).

To prove the assertion we distinguish the two cases $V\cap\partial\Omega\subset\Gamma_0\backslash Z$ and $V\cap\partial\Omega\subset\Gamma_1\backslash Z$.

In the first case we have $u\circ\psi\equiv 0$ on U. Differentiating this identity we obtain $(u_x\circ\psi)'\cdot D[\psi]\equiv 0$ on U. Evaluating at 0 we see that $u_x(x)$ is orthogonal to T_x, i.e., $u_x(x)=\alpha\nu(x)$ for some $\alpha\in\mathbb{R}$. Multiplication by $\nu(x)'$ yields $\alpha=(\partial u/\partial\nu)(x)$ and thus,

$$u_x(x)=\frac{\partial u}{\partial\nu}(x)\cdot\nu(x). \tag{34}$$

Differentiating the identity $(u_x\circ\psi)'\cdot D[\psi]\equiv 0$ a second time and evaluating at 0 we obtain, using (34),

$$D[\psi](0)'\cdot u_{xx}(x)\cdot D[\psi](0)=-\frac{\partial u}{\partial\nu}(x)\cdot\sum_{k=1}^{n}\nu_k(x)\,D^2[\psi_k](0). \tag{35}$$

Now let Φ denote the $n\times n$-matrix given by $\Phi:=(D[\psi](0)\mid\nu(x))$. Since Φ is *orthogonal* we obtain

$$\begin{aligned}\Delta u(x)&=\operatorname{trace}[\Phi' u_{xx}(x)\Phi]\\&=\operatorname{trace}[D[\psi](0)'\,u_{xx}(x)\,D[\psi](0)]+\nu(x)'\,u_{xx}(x)\,\nu(x).\end{aligned}$$

Together with (30), (34), and (35) it follows that

$$R[u](x)=\left[\frac{\partial u}{\partial\nu}(x)\right]^2\cdot\sum_{k=1}^{n}\nu_k(x)\cdot\operatorname{trace}(D^2[\psi_k](0)),$$

and the latter sum equals $\operatorname{trace}(S_x)=(n-1)\,H(x)$, as can easily be derived from (31) and the orthonormality of the columns of $D[\psi](0)$.

In the second case, $V\cap\partial\Omega\subset\Gamma_1\backslash Z$, the identity $(u_x\circ\psi)'\cdot(\nu\circ\psi)\equiv 0$ holds on U. Differentiation and evaluation at 0 provides

$$\nu(x)'\cdot u_{xx}(x)\cdot D[\psi](0)=-u_x(x)'\cdot D[\nu\circ\psi](0). \tag{36}$$

Moreover, the equation $u_x(x)'\,\nu(x)=0$ implies $u_x(x)\in T_x$ and thus, $u_x(x)=D[\psi](0)\cdot w$ for some $w\in\mathbb{R}^{n-1}$. Multiplication by $D[\psi](0)'$ yields $w=D[\psi](0)'\cdot u_x(x)$ and therefore, $u_x(x)=D[\psi](0)\cdot D[\psi](0)'\cdot u_x(x)$.

Together with (30), the equations $(\partial u/\partial \nu)(x)=0$, (36), and (31), it follows that

$$\begin{aligned} R[u](x) &= \nu(x)^t \cdot u_{xx}(x) \cdot u_x(x) = -u_x(x)^t \cdot D[\nu \circ \psi](0) \cdot D[\psi](0)^t \cdot u_x(x) \\ &= u_x(x)^t S_x u_x(x) \end{aligned}$$

which equals $N(x, u_x(x)) \cdot u_x(x)^t u_x(x)$ if $u_x(x) \neq 0$, and 0 otherwise. ∎

LEMMA 3. *With the assumptions of Lemma* 2,

$$\int_{\partial\Omega} R[u]\, d\sigma \leqslant 2F_0 K_1 \, \|\Delta u\|_{2,\Omega} \, \|L[u]\|_{2,\Omega} + F_1 K_1^2 \, \|L[u]\|_{2,\Omega}^2.$$

Proof. Lemma 2, the estimate $N(x, v) \leqslant P(x)$ $(x \in \Gamma_1 \backslash Z,\ v \in T_x \backslash \{0\})$, and (27) provide

$$R[u] \leqslant (f^t \cdot \nu) \left(\frac{\partial u}{\partial \nu}\right)^2 \qquad \text{a.e. on } \Gamma_0,$$

$$R[u] \leqslant -(f^t \cdot \nu) \cdot (u_x^t \cdot u_x) \qquad \text{a.e. on } \Gamma_1.$$

Regarding that $u_x = (\partial u/\partial \nu) \cdot \nu$ almost everywhere on Γ_0 (see (34)) and $\partial u/\partial \nu = 0$ almost everywhere on Γ_1, we therefore obtain

$$R[u] \leqslant 2(f^t \cdot u_x) \frac{\partial u}{\partial \nu} - (f^t \cdot \nu)(u_x^t \cdot u_x) \qquad \text{a.e. on } \partial\Omega. \tag{37}$$

Now we make use of a famous identity found by Payne and Weinberger [15, p. 553, formula (2.4)] which holds for $u \in C_2(\bar{\Omega})$ and is proved by two steps of partial integration:

$$\begin{aligned} &\int_{\partial\Omega} \left[2(f^t \cdot u_x) \frac{\partial u}{\partial \nu} - (f^t \cdot \nu)(u_x^t \cdot u_x)\right] d\sigma \qquad (38) \\ &\quad = 2 \int_\Omega (f^t \cdot u_x)\, \Delta u \, dx + \int_\Omega u_x^t \cdot [-(\operatorname{div} f) \cdot I + D[f] + D[f]^t] \cdot u_x \, dx. \end{aligned}$$

Equations (37), (38), (28) and Schwarz's inequality show that

$$\int_{\partial\Omega} R[u]\, d\sigma \leqslant 2F_0 \, \|u_x\|_{2,\Omega} \, \|\Delta u\|_{2,\Omega} + F_1 \, \|u_x\|_{2,\Omega}^2,$$

and the assertion follows by use of (23). ∎

LEMMA 4. $\|\Delta u\|_{2,\Omega} \leqslant \kappa \, \|L[u]\|_{2,\Omega}$ *for u as in Lemma* 2.

Proof. First we consider the case where, with $\bar{c}$ and $\underline{c}$ from (20), $\bar{c}+\underline{c}\leqslant 0$, and thus, $\frac{1}{2}(\bar{c}-\underline{c})\leqslant -\underline{c}$ and $\|c\|_{\infty,\Omega}\leqslant -\underline{c}$. Consequently, $\|\Delta u\|_{2,\Omega}=\|L[u]-cu\|_{2,\Omega}\leqslant \|L[u]\|_{2,\Omega}-\underline{c}\,\|u\|_{2,\Omega}$, and the assertion follows by use of (18).

In the case $\bar{c}+\underline{c}>0$ we obtain, with $\mu:=\frac{1}{2}(\bar{c}+\underline{c})$,

$$\int_\Omega (-\Delta u+\mu u)^2\,dx=\int_\Omega (\Delta u)^2\,dx-2\mu\int_\Omega u\cdot\Delta u\,dx+\mu^2\int_\Omega u^2\,dx.$$

Applying partial integration to the second term on the right-hand side and using the boundary condition for u we derive $\|-\Delta u+\mu u\|_{2,\Omega}\geqslant\|\Delta u\|_{2,\Omega}$ and thus,

$$\|\Delta u\|_{2,\Omega}\leqslant\|L[u]+(\mu-c)u\|_{2,\Omega}\leqslant\|L[u]\|_{2,\Omega}+\|\mu-c\|_{\infty,\Omega}\cdot\|u\|_{2,\Omega}.$$

Observing that $\|\mu-c\|_{\infty,\Omega}\leqslant\frac{1}{2}(\bar{c}-\underline{c})$ and using (18) we obtain the assertion. ▮

Lemmata 1, 3, and 4 show that the inequality asserted in Theorem 3 holds for $u\in C_2(\bar{\Omega})$ satisfying $B[u]=0$ a.e. on $\partial\Omega$. Since the set of these u is dense in $H_2^B(\Omega)$ with respect to the norm in $H_2(\Omega)$, the assertion holds for $u\in H_2^B(\Omega)$. ▮

5. Auxiliary Lemmata

In this intermediate section, we present two lemmata which will be needed in the following. The first is concerned with the solvability of boundary value problems in $H_2^B(\Omega)$. We will call the triple $(\Omega,\Gamma_0,\Gamma_1)$ *regular* if, for some $\sigma>0$, the boundary value problem

$$u\in H_2^B(\Omega),\qquad -\Delta u+\sigma u=r \text{ on } \Omega \tag{39}$$

has a solution for a set of functions r which is *dense* in $L_2(\Omega)$.

Some general examples of regular $(\Omega,\Gamma_0,\Gamma_1)$ will be given after the following lemma.

Lemma 5. *Suppose that* (6) *holds and* $(\Omega,\Gamma_0,\Gamma_1)$ *is regular. Then, the boundary value problem* $u\in H_2^B(\Omega)$, $L[u]=r$ *on* Ω, *has a unique solution for each* $r\in L_2(\Omega)$.

Proof. Integration by parts shows that, for $u\in C_2(\bar{\Omega})$ satisfying $B[u]=0$ on $\partial\Omega$, $\|-\Delta u+\sigma u\|_{2,\Omega}^2\geqslant\sigma^2\,\|u\|_{2,\Omega}^2$. Therefore, $\|u\|_{2,\Omega}\leqslant\sigma^{-1}\,\|-\Delta u+\sigma u\|_{2,\Omega}$ for $u\in H_2^B(\Omega)$. Using (6) one easily derives that also the second and the third estimates in (6) hold with $-\Delta+\sigma$ in place of L

(and with new constants $\tilde{K}_1$, $\tilde{K}_2$). Thus, the inverse operator $(-\Delta+\sigma)^{-1}$ is bounded and, due to the regularity assumption, densely defined. Consequently, it may be extended to a bounded linear operator $(-\Delta+\sigma)^{-1}$: $L_2(\Omega) \to H_2^B(\Omega)$.

The boundary value problem $u \in H_2^B(\Omega)$, $L[u]=r$ on Ω, is therefore equivalent to the Fredholm equation

$$u \in L_2(\Omega), \qquad u = Tu + (-\Delta+\sigma)^{-1} r, \tag{40}$$

where $Tu := (-\Delta+\sigma)^{-1} [(\sigma-c)u]$. Since $\sigma - c$ is bounded on Ω, $T: L_2(\Omega) \to H_2^B(\Omega)$ is a bounded linear operator and therefore compact as an operator from $L_2(\Omega)$ into itself. Thus, Fredholm's alternative shows that (40) is uniquely solvable, since the homogeneous problem ($r \equiv 0$) has only the trivial solution due to (6). ∎

EXAMPLES. (1) $(\Omega, \Gamma_0, \Gamma_1)$ is regular if $\partial\Omega$ is a global C_2-hypersurface and, moreover, $\partial\Omega = \Gamma_0$ or $\partial\Omega = \Gamma_1$. See [7, Lemma 18.2 (and Lemma 19.1 in connection with problem (6) after Theorem 19.4) for the case $\partial\Omega = \Gamma_0$, and Theorem 19.3 for both cases $\partial\Omega = \Gamma_0$ and $\partial\Omega = \Gamma_1$].

(2) The results just mentioned may be carried over to "regular" mixed boundary value problems where each connected component C of $\partial\Omega$ satisfies $C \subset \Gamma_0$ or $C \subset \Gamma_1$, and to domains with a $C_{1,1}$-boundary (which may locally be parametrized by a C_1-function with Lipschitz-continuous first derivatives).

(3) Let Ω, Γ_0, Γ_1 have the property that the eigenvalue problem $\varphi \in H_2^B(\Omega)$, $-\Delta\varphi = \lambda\varphi$ on Ω, has a complete system $(\varphi_j)_{j \in \mathbb{N}}$ of orthonormal eigenfunctions $\varphi_j \in H_2^B(\Omega)$. Then, $(\Omega, \Gamma_0, \Gamma_1)$ is regular since the set of all functions $r = \sum_{j=1}^{N} \alpha_j \varphi_j$ (with $N \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$) is dense in $L_2(\Omega)$, and the boundary value problem (39) is solved, for such r, by $u := \sum_{j=1}^{N} (\lambda_j + \sigma)^{-1} \alpha_j \varphi_j$, with $(\lambda_j)_{j \in \mathbb{N}}$ denoting the sequence of corresponding eigenvalues and $\sigma \neq -\lambda_j$ for all j. (Compare [12, Chap. 3, Section 9].)

In particular, this assumption holds for many domains with *known* eigenfunctions, such as rectangles (in arbitrary dimension) with each side belonging completely either to Γ_0 or to $\bar{\Gamma}_1$, circular disks, balls and shells, circular sectors (in two dimensions) and circular cones (in higher dimensions) with each "side" (including the spherical part) belonging completely either to Γ_0 or to $\bar{\Gamma}_1$ and with interior angle $\vartheta \in (0, \pi]$. If $n = 2$ and Γ_0 and $\bar{\Gamma}_1$ "meet" at the angular point, ϑ must further be restricted to $(0, \pi/2]$. Moreover, each cylinder $\Omega := \tilde{\Omega} \times (0, T) \subset \mathbb{R}^n$, with $\tilde{\Omega} \subset \mathbb{R}^{n-1}$ denoting a domain of one of the types considered above (for instance, a domain with C_2-smooth boundary), has the desired properties.

(4) Suppose that $\partial\Omega = \Gamma_0$ and that $\bar{\Omega}$ may be mapped by a $C_{1,1}$-diffeomorphism ϕ (i.e., a C_1-diffeomorphism with Lipschitz-continuous first derivatives) onto $\bar{\Omega}_0$, with Ω_0 denoting a domain such that $(\Omega_0, \partial\Omega_0, \emptyset)$ is regular. Moreover, let (26) hold (with $\Gamma_1 = \emptyset$) and let the maximal principal curvature be essentially bounded from above on $\partial\Omega$. Then, $(\Omega, \partial\Omega, \emptyset)$ is regular. This can be seen as follows: The boundary value problem (39) is equivalent to the following problem for $v := u \circ \phi^{-1}$, $s := r \circ \phi^{-1}$,

$$v \in H_2^B(\Omega_0),$$

$$L_0[v] := -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 v}{\partial x_i \, \partial x_j} - \sum_{i=1}^{n} b_i \frac{\partial v}{\partial x_i} + \sigma v = s \qquad \text{on } \Omega_0, \tag{41}$$

where $A = (a_{ij}) := (D[\phi] \cdot D[\phi]') \circ \phi^{-1}$, $b = (b_i) := (\Delta\phi) \circ \phi^{-1}$. Without going into details we state that estimates of the type (6) may also be derived with $L^{(\tau)} := -\Delta + \sigma + \tau(L_0 + \Delta - \sigma)$ $(0 \leqslant \tau \leqslant 1)$ and Ω_0 in place of L and Ω, with uniform constants for $\tau \in [0, 1]$. Using this a priori estimate and the regularity of $(\Omega_0, \partial\Omega_0, \emptyset)$, and applying the usual continuation process along $\tau \in [0, 1]$ (compare [12, p. 111 ff]) we obtain that problem (41) has a unique solution and thus, $(\Omega, \partial\Omega, \emptyset)$ is regular.

For example, $(\Omega, \partial\Omega, \emptyset)$ is therefore regular for parallelepipeds (in arbitrary dimension), triangles (in two dimensions), and cones, which may be mapped $C_{1,1}$-diffeomorphically onto rectangles, circular sectors, and spherical cones, respectively.

Our second auxiliary lemma deals with *weak differential inequalities* and *inverse-positivity*:

LEMMA 6. *Suppose that some $z \in H_2(\Omega)$ exists such that, for some $\rho \in \mathbb{R}$,*

$$\operatorname*{ess\,inf}_{\Omega} z > 0, \qquad \operatorname*{ess\,inf}_{\Omega} (L[z] + \rho z) > 0, \qquad B[z] \geqslant 0 \text{ a.e. on } \partial\Omega. \tag{42}$$

Then, for each $w \in H_2(\Omega)$, the inequalities $L[w] + \rho w \geqslant 0$ (a.e. on Ω) and $B[w] \geqslant 0$ (a.e. on $\partial\Omega$) together imply $w \geqslant 0$ (a.e. on Ω). In other words, $(L + \rho, B)$ is inverse-positive on $H_2(\Omega)$.

Proof. The lemma could be proved, without restriction on the dimension n, by use of the generalized maximum principle presented in [5, Theorem 1]. Here, we will give a simpler direct proof which follows the lines of the proof given in [21, Theorem 3.8, p. 189] for the one-dimensional case and which uses the *boundedness* (resp. the continuity) of z and w provided by the embedding $H_2(\Omega) \subsetneq C_0(\bar{\Omega})$ which holds for $n \leqslant 3$.

Since $z, w \in C_0(\bar{\Omega})$ and $z(x) > 0$ $(x \in \bar{\Omega})$ due to (42), there exist numbers $\lambda \in \mathbb{R}$ such that $w + \lambda z \geqslant 0$ on $\bar{\Omega}$. Let λ_0 denote the smallest of these numbers and assume for contradiction that $\lambda_0 > 0$. For $\lambda \in [0, \lambda_0)$, let $u_\lambda := w + \lambda z$ and $u_\lambda^+(x) := \max\{0, u_\lambda(x)\}$, $u_\lambda^-(x) := \max\{0, -u_\lambda(x)\}$ for $x \in \bar{\Omega}$. Using (42) and the inequalities assumed for w one easily derives, with $\varepsilon := \operatorname{ess\,inf}_\Omega(L[z] + \rho z) > 0$,

$$L[u_\lambda] + \rho u_\lambda \geqslant \lambda\varepsilon \text{ a.e. in } \Omega, \qquad \frac{\partial u_\lambda}{\partial \nu} \geqslant 0 \text{ a.e. on } \Gamma_1,$$

$$u_\lambda^-(x) = 0 \text{ for } x \in \Gamma_0,$$

$$\|u_\lambda^-\|_{\infty,\Omega} \leqslant (\lambda_0 - \lambda)\,\|z\|_{\infty,\Omega}, \qquad \int_\Omega u_\lambda^-(x)\,dx > 0. \tag{43}$$

Moreover, Lemma 7.6 in [8, p. 152] shows that u_λ^+ and u_λ^- have weak first derivatives (gradients) $(u_\lambda^+)_x$ and $(u_\lambda^-)_x$ and that, for almost all $x \in \Omega$,

$$\begin{aligned}(u_\lambda^+)_x(x) &= \begin{Bmatrix} (u_\lambda)_x(x) & \text{if } u_\lambda(x) > 0 \\ 0 & \text{if } u_\lambda(x) \leqslant 0 \end{Bmatrix}, \\ (u_\lambda^-)_x(x) &= \begin{Bmatrix} 0 & \text{if } u_\lambda(x) \geqslant 0 \\ -(u_\lambda)_x(x) & \text{if } u_\lambda(x) < 0 \end{Bmatrix}.\end{aligned} \tag{44}$$

In particular, these relations show that u_λ^+ and u_λ^- belong to $H_1(\Omega)$. Now let Q denote the symmetric bilinear form associated to $L + \rho$, i.e.,

$$Q(u, v) := \int_\Omega [u_x(x)^t v_x(x) + (c(x) + \rho)\, u(x)\, v(x)]\, dx \qquad (u, v \in H_1(\Omega)).$$

Relation (44) shows that $Q(u_\lambda^+, u_\lambda^-) = 0$ and thus, since $u_\lambda = u_\lambda^+ - u_\lambda^-$,

$$Q(u_\lambda, u_\lambda^-) = -Q(u_\lambda^-, u_\lambda^-) \leqslant -(\underline{c} + \rho) \int_\Omega (u_\lambda^-)^2\, dx, \tag{45}$$

with $\underline{c}$ denoting a constant lower bound for c. On the other hand, partial integration and the first three relations in (43) imply

$$\begin{aligned} Q(u_\lambda, u_\lambda^-) &= \int_\Omega (L[u_\lambda] + \rho u_\lambda)\, u_\lambda^-\, dx + \int_{\Gamma_1} \frac{\partial u_\lambda}{\partial \nu}\, u_\lambda^-\, d\sigma \\ &\geqslant \lambda\varepsilon \int_\Omega u_\lambda^-\, dx. \end{aligned} \tag{46}$$

Since $u_\lambda^- \geqslant 0$, (45) and (46) show that $\underline{c} + \rho \leqslant 0$ and

$$\lambda\varepsilon \int_\Omega u_\lambda^-\, dx \leqslant |\underline{c} + \rho| \int_\Omega (u_\lambda^-)^2\, dx \leqslant |\underline{c} + \rho| \cdot \|u_\lambda^-\|_{\infty,\Omega} \cdot \int_\Omega u_\lambda^-\, dx.$$

Using the two final relations in (43) we therefore obtain $\lambda\varepsilon \leqslant |\underline{c}+\rho| \cdot (\lambda_0-\lambda) \cdot \|z\|_{\infty,\Omega}$ which yields a contradiction for λ sufficiently close to λ_0. ∎

6. Pointwise Bounds

The results of Sections 2, 3, and 4 provide the estimate (7) with

$$K := C_0 K_0 + C_1 K_1 + C_2 K_2. \tag{47}$$

Before turning to the general estimate (3) we show how, in the case where B is the *Dirichlet*-boundary operator ($\partial\Omega = \Gamma_0$), a constant K satisfying (7) may be calculated in a simpler way, in particular, without computing a function f satisfying (27). This simplification is achieved by application of the results of Sections 2 through 5 to a "simpler" domain containing Ω.

Theorem 4. *Let $\partial\Omega = \Gamma_0$ and suppose that* (18) *and* (20) *hold with constants K_0, $\underline{c}$, $\bar{c}$. Moreover, let some convex domain $\hat{\Omega} \supset \Omega$ be given such that $\partial\hat{\Omega}$ is a piecewise C_2-hypersurface and $(\hat{\Omega}, \partial\hat{\Omega}, \emptyset)$ is regular in the sense defined in Section 5. Let $\mu_0 > 0$ denote a lower bound for the smallest eigenvalue of $-\Delta$ on $\hat{\Omega}$ with respect to Dirichlet boundary conditions. Finally, suppose that* (5) *holds with $\hat{\Omega}$ in place of Ω and with constants $\hat{C}_0$, $\hat{C}_1$, $\hat{C}_2$. Then, for each $\tau \geqslant 0$,* (7) *is true for*

$$K := \left[\frac{1}{\mu_0+\tau}\,\hat{C}_0 + \psi(\tau)\cdot\hat{C}_1 + \left(1+\frac{1}{2}\,\frac{\bar{c}-\underline{c}}{\mu_0+\tau}\right)\hat{C}_2\right]\cdot[1+|\tau-\underline{c}|\;K_0], \tag{48}$$

where $\psi(\tau) := \sqrt{\mu_0}/(\mu_0+\tau)$ if $\tau \leqslant \mu_0$ and $\psi(\tau) := 1/(2\sqrt{\tau})$ otherwise.

Remarks. (1) In (48), $\tau \geqslant 0$ is a free parameter. Since the first factor in brackets is monotonically decreasing in τ, only choices $\tau \geqslant \max\{0, \underline{c}\}$ are efficient with regard to a "small" value for K.

(2) In particular, one may choose $\hat{\Omega}$ to be a *ball* (of radius R) containing Ω, and $\mu_0 := 2nR^{-2}$, or a *rectangle* (with sidelengths $L_1, ..., L_n$) containing Ω, and $\mu_0 := \pi^2 \cdot \sum_{i=1}^{n} L_i^{-2}$. For these choices, the examples at the end of Section 2 show how to compute the constants $\hat{C}_0$, $\hat{C}_1$, $\hat{C}_2$ needed in (48). Alternatively, one may use part (b) of Theorem 1 (so that, in particular, $\hat{C}_0 = 0$). Thus, K_0 is the only term in (48) which is not completely explicit; see Section 3 for its computation.

Proof of Theorem 4. It suffices to prove the assertion for $\tau > 0$. Let $\hat{c} \in L_\infty(\hat{\Omega})$ denote some extension of c to $\hat{\Omega}$ satisfying $\underline{c} \leqslant \hat{c}(x) \leqslant \bar{c}$ for $x \in \hat{\Omega}$.

Moreover, let $\hat{L}[u] := -\Delta u + (\hat{c}(\cdot) - \underline{c} + \tau)u$ for $u \in H_2(\hat{\Omega})$. Since $\hat{c} - \underline{c}$ is nonnegative on $\hat{\Omega}$, all eigenvalues of $\hat{L}$ on $H_2^{\hat{B}}(\hat{\Omega})$ (with $\hat{B}$ denoting the Dirichlet boundary operator for $\hat{\Omega}$) are bounded from below by $\mu_0 + \tau$. Consequently (compare (22)),

$$\|u\|_{2,\hat{\Omega}} \leqslant \frac{1}{\mu_0 + \tau} \|\hat{L}[u]\|_{2,\hat{\Omega}} \qquad \text{for} \quad u \in H_2^{\hat{B}}(\hat{\Omega}).$$

Using Theorem 2, Corollary 2, and the convexity of $\hat{\Omega}$, one easily derives

$$\|u_x\|_{2,\hat{\Omega}} \leqslant \psi(\tau) \|\hat{L}[u]\|_{2,\hat{\Omega}},$$

$$\|u_{xx}\|_{2,\hat{\Omega}} \leqslant \left(1 + \frac{1}{2} \frac{\bar{c} - \underline{c}}{\mu_0 + \tau}\right) \|\hat{L}[u]\|_{2,\hat{\Omega}} \qquad (u \in H_2^{\hat{B}}(\hat{\Omega})).$$

Thus, with $\hat{K}$ denoting the first of the two factors in brackets in (48),

$$\|u\|_{\infty,\hat{\Omega}} \leqslant \hat{K} \|\hat{L}[u]\|_{2,\hat{\Omega}} \qquad \text{for} \quad u \in H_2^{\hat{B}}(\hat{\Omega}). \tag{49}$$

Moreover, the assumptions of Lemma 5 are satisfied with $\hat{\Omega}$, $\hat{L}$ in place of Ω , L.

Now let some $u \in H_2^B(\Omega)$ be given and define $r \in L_2(\hat{\Omega})$ by $r(x) := |L[u](x) + (\tau - \underline{c})\, u(x)|$ for $x \in \Omega$, $r(x) := 0$ for $x \in \hat{\Omega} \backslash \Omega$. Due to Lemma 5, the boundary value problem

$$v \in H_2^{\hat{B}}(\hat{\Omega}), \qquad \hat{L}[v] = r \text{ on } \hat{\Omega} \tag{50}$$

has a unique solution. Applying (49) to v we obtain

$$\begin{aligned}\|v\|_{\infty,\hat{\Omega}} &\leqslant \hat{K} \|r\|_{2,\hat{\Omega}} = \hat{K} \|L[u] + (\tau - \underline{c})u\|_{2,\Omega} \\ &\leqslant \hat{K}(\|L[u]\|_{2,\Omega} + |\tau - \underline{c}| \cdot \|u\|_{2,\Omega}) \\ &\leqslant \hat{K}(1 + |\tau - \underline{c}|\, K_0) \cdot \|L[u]\|_{2,\Omega}.\end{aligned}$$

Here, we used (18) in the final step. It remains to show that $\|u\|_{\infty,\Omega} \leqslant \|v\|_{\infty,\hat{\Omega}}$ which we carry out using Lemma 6.

With $\hat{\Omega}$ and $\hat{L}$ in place of Ω and L, (42) obviously holds for $z \equiv 1$ (and $\rho = 0$), since $\tau > 0$. Equation (50), the nonnegativity of r on $\hat{\Omega}$, and Lemma 6 therefore imply $v \geqslant 0$ on $\hat{\Omega}$ and thus, on $\partial\Omega$. Since u vanishes on $\partial\Omega$, it follows that

$$w_i(x) \geqslant 0 \qquad (x \in \partial\Omega;\ i = 1, 2) \tag{51}$$

for $w_1 := v\,|_{\Omega} - u$ and $w_2 := v\,|_{\Omega} + u$. Moreover, (50) and the definition of r imply

$$L[w_i] + (\tau - \underline{c})\, w_i \geqslant 0 \qquad \text{on} \quad \Omega\ (i = 1, 2). \tag{52}$$

Obviously, (42) holds for $z \equiv 1$ and $\rho = \tau - \underline{c}$. Lemma 6, (51), and (52) therefore provide $w_i \geqslant 0$ on Ω $(i = 1, 2)$; i.e., $|u(x)| \leqslant v(x)$ for $x \in \bar{\Omega}$, which implies $\|u\|_{\infty, \Omega} \leqslant \|v\|_{\infty, \Omega} \leqslant \|v\|_{\infty, \hat{\Omega}}$. ∎

Now we return to the general boundary operator (17) (with Γ_1 not necessarily empty). Next we present our main theorem concerned with the general estimate (3).

THEOREM 5. *Suppose that* (6) holds and $(\Omega, \Gamma_0, \Gamma_1)$ *is regular in the sense defined in Section* 5. *Let some* $z \in H_2(\Omega)$ *be given which satisfies*

$$\operatorname*{ess\,inf}_{x \in \Omega} z(x) > 0, \qquad \operatorname*{ess\,sup}_{x \in \Omega} \Delta z(x) < \infty, \qquad B[z] \geqslant 1 \text{ a.e. on } \partial\Omega \tag{53}$$

and define $\mu := \max\{0, \operatorname{ess\,sup}_{x \in \Omega}\{\Delta z(x)/z(x) - c(x)\}\}$. *Then,* (3) *holds with* K *satisfying* (7) *and* $K_B := \|z\|_{\infty, \Omega} + \mu K \|z\|_{2, \Omega}$.

EXAMPLE. If $\partial\Omega = \Gamma_0$, (53) holds for $z \equiv 1$. If (6) holds and $(\Omega, \partial\Omega, \varnothing)$ is regular, (3) is therefore true with K satisfying (7), and

$$K_B := 1 + \max\{0, -\underline{c}\} K \sqrt{\operatorname{meas}(\Omega)}$$

with $\underline{c}$ from (20). In particular, $K_B = 1$ if $\underline{c} \geqslant 0$. In that case, (3) therefore shows that

$$\|u\|_{\infty, \Omega} \leqslant \|u\|_{\infty, \partial\Omega} \qquad \text{for each } u \in H_2(\Omega) \text{ satisfying } L[u] = 0 \text{ on } \Omega,$$

which is in fact a special formulation of the maximum principle.

In the general case (where $\Gamma_1 \neq \varnothing$) the problem of finding a function z satisfying (53) can be solved by considerations similar to those made in Section 4 (see, in particular, Remark 2 after Corollary 2) to construct a function f satisfying (27).

Proof of Theorem 5. It suffices to prove the assertion for each $\rho \in (\mu, \infty)$ in place of μ. Thus, let $\rho > \mu$ be fixed. Conditions (53) and the definition of μ show that (42) is satisfied. Consequently, $(L + \rho, B)$ is inverse-positive on $H_2(\Omega)$ due to Lemma 6. In particular, $L + \rho$ is invertible on $H_2^B(\Omega)$. Thus, since its spectrum is discrete, $\|u\|_{2, \Omega} \leqslant \tilde{K}_0 \|L[u] + \rho u\|_{2, \Omega}$ $(u \in H_2^B(\Omega))$ for some constant $\tilde{K}_0$. Using (6) one easily derives that also the second and the third estimates in (6) hold with $L + \rho$ in place of L (and with new constants $\tilde{K}_1$, $\tilde{K}_2$).

Now let some $u \in \hat{H}_2^B(\Omega)$ be given. Due to Lemma 5 (with $L + \rho$ in place of L), the boundary value problem

$$v \in H_2^B(\Omega), \qquad L[v] + \rho v = L[u] + \rho u \qquad \text{on } \Omega \tag{54}$$

has a unique solution. Applying (7) to v we obtain $\|v\|_{\infty,\Omega} \leqslant K\,\|L[v]\|_{2,\Omega} \leqslant K\,\|L[u]\|_{2,\Omega} + \rho K\,\|u-v\|_{2,\Omega}$ and thus,

$$\|u\|_{\infty,\Omega} \leqslant K\,\|L[u]\|_{2,\Omega} + \|u-v\|_{\infty,\Omega} + \rho K\,\|u-v\|_{2,\Omega}. \tag{55}$$

Let $\delta := \|B[u]\|_{\infty,\partial\Omega}$. The definition of μ, (53), and (54) show that $L[w_i] + \rho w_i \geqslant 0$ on Ω, $B[w_i] \geqslant 0$ on $\partial\Omega$ $(i = 1, 2)$, where $w_1 := \delta z - (u - v)$, $w_2 := \delta z + (u - v)$. Thus, since $(L + \rho, B)$ is inverse-positive, $w_i \geqslant 0$ on Ω $(i = 1, 2)$; i.e.,

$$|u(x) - v(x)| \leqslant \delta z(x) \qquad \text{for} \quad x \in \Omega. \tag{56}$$

The estimates (55), (56), and the choice of δ prove the assertion (with ρ in place of μ). ∎

7. An Existence and Inclusion Result for Nonlinear Problems

In this section, we will show how the explicit estimates derived in the preceding sections can be used to obtain existence results for nonlinear boundary value problems of the form (1), in combination with error bounds for approximate solutions. The general framework of the algorithm described below is due to Schröder (see [22, 23], for instance). For ordinary boundary value problems, this algorithm has been realized in computer programs and applied successfully to several examples; see [4, 9, 17, 18]. Numerical examples of elliptic boundary value problems are given in [20].

Suppose that an *approximate solution* $\omega \in H_2(\Omega)$ of problem (1) has been computed such that $B[\omega] - s \in L_\infty(\partial\Omega)$. For that purpose, finite-element methods (with C_1-elements) or difference methods in connection with interpolation techniques are reasonable.

Moreover, let the *defect estimates* (2) hold with constants δ and δ_B, and let constants K and K_B be given such that (3) is satisfied, where L is defined by (4) with $c(x) := (\partial F/\partial U)(x, \omega(x))$ $(x \in \bar{\Omega})$.

Finally, let $G : [0, \infty) \to [0, \infty)$ denote a monotonically nondecreasing function satisfying

$$|F(x, \omega(x) + y) - F(x, \omega(x)) - c(x)\,y)| \leqslant G(|y|) \qquad (x \in \bar{\Omega},\ y \in \mathbb{R}). \tag{57}$$

Since $\partial F/\partial U$ is continuous on $\bar{\Omega} \times \mathbb{R}$, G may be chosen such that

$$G(t) = o(t) \qquad \text{for} \quad t \to 0. \tag{58}$$

In most practical cases, such a *majorizing* function G can easily be calculated if constant upper and lower bounds for ω are known.

THEOREM 6. *In addition to the assumptions made above, let* (6) *hold and let* $(\Omega, \Gamma_0, \Gamma_1)$ *be regular in the sense defined in Section* 5. *Moreover, suppose that some* $\alpha \geqslant 0$ *exists such that*

$$\alpha - K\sqrt{\operatorname{meas}(\Omega)} \cdot G(\alpha) \geqslant K\delta + K_B \delta_B. \tag{59}$$

Then, there exists a solution $U \in H_2(\Omega)$ *of problem* (1) *satisfying*

$$\|U - \omega\|_{\infty, \Omega} \leqslant \alpha. \tag{60}$$

The boundary condition is satisfied in the trace-sense and, a fortiori, in the sense that $U - \bar{s} \in H_2^B(\Omega)$ *for the given function* $\bar{s}$ *satisfying* $B[\bar{s}] = s$.

Due to (58), the crucial condition (59) is satisfied for some "small" α, if δ and δ_B are sufficiently small, i.e., if the approximate solution ω has been calculated with *sufficient accuracy*.

Proof of Theorem 6. Obviously, problem (1) is equivalent to the following boundary value problem for the *error* $u = U - \omega$:

$$\begin{aligned} L[u] + f(\cdot, u) &= -d \qquad \text{on } \Omega, \\ B[u] = s - B[\omega] &= B[\bar{s} - \omega] \qquad \text{on } \Omega, \end{aligned} \tag{61}$$

where $f(x, y) := F(x, \omega(x) + y) - F(x, \omega(x)) - c(x)\, y$ and $d := -\Delta\omega + F(\cdot, \omega) \in L_2(\Omega)$. Thus, it suffices to show that problem (61) has a solution $u \in H_2(\Omega)$ satisfying $\|u\|_{\infty, \Omega} \leqslant \alpha$.

Lemma 5 provides the existence of the inverse operator L^{-1}: $L_2(\Omega) \to H_2^B(\Omega)$ mapping each $r \in L_2(\Omega)$ onto the unique solution of the boundary value problem $v \in H_2^B(\Omega)$, $L[v] = r$ on Ω. Moreover, L^{-1} is bounded due to (6). Treating the boundary condition in (61) appropriately we see that problem (61) is solved by each fixed-point $u \in C_0(\bar{\Omega})$ of the operator T: $C_0(\bar{\Omega}) \to C_0(\bar{\Omega})$ defined by

$$Tu := \bar{s} - \omega - L^{-1}(L[\bar{s} - \omega] + d + f(\cdot, u)). \tag{62}$$

Since, due to our assumption $n \leqslant 3$, the embedding $H_2(\Omega) \hookrightarrow C_0(\bar{\Omega})$ is compact (see [1, Theorem 6.2, p. 144]), T is a continuous and compact operator. Thus, the assertion follows from Schauder's fixed-point theorem, if we show that

$$TD \subset D \qquad \text{for} \quad D := \{u \in C_0(\bar{\Omega}) : \|u\|_{\infty, \Omega} \leqslant \alpha\}.$$

For $u \in D$, the estimate (57) and the monotonicity of G imply

$|f(x, u(x))| \leqslant G(|u(x)|) \leqslant G(\alpha) \quad (x \in \bar{\Omega})$ and thus, $\|f(\cdot, u)\|_{2,\Omega} \leqslant \sqrt{\text{meas}(\Omega)} \cdot G(\alpha)$. Together with (3), (62), (2), and (59) it follows that

$$\begin{aligned} \|Tu\|_{\infty,\Omega} &\leqslant K\,\|L[Tu]\|_{2,\Omega} + K_B\,\|B[Tu]\|_{\infty,\partial\Omega} \\ &= K\,\|d + f(\cdot, u)\|_{2,\Omega} + K_B\,\|s - B[\omega]\|_{\infty,\partial\Omega} \\ &\leqslant K \cdot [\delta + \sqrt{\text{meas}(\Omega)} \cdot G(\alpha)] + K_B \delta_B \leqslant \alpha. \quad \blacksquare \end{aligned}$$

If one is interested in *classical* solutions of problem (1), one may use the well-known "bootstrapping" technique and regularity theorems for linear problems. Suppose, for instance, that the nonlinearity F is α-Hölder-continuous with respect to x, uniformly on each compact subset of $\Omega \times \mathbb{R}$ (and that, as already assumed for Theorem 6, $\partial F/\partial U$ is continuous on $\bar{\Omega} \times \mathbb{R}$). Moreover, let $U \in H_2(\Omega)$ denote a solution of problem (1) (provided, for example, by Theorem 6). Then, $U \in C_{0,1/2}(\bar{\Omega})$ (i.e., U is uniformly $\frac{1}{2}$-Hölder-continuous) due to Sobolev's embedding theorem [1, Theorem 5.4, pp. 97, 98]. Consequently, $F(\cdot, U) \in C_{0,\beta}(\Omega)$, where $\beta := \min\{\alpha, \frac{1}{2}\}$. Regarding $F(\cdot, U)$ as inhomogenity in the differential equation in (1) we now can conclude by use of regularity theorems (for instance, [12, Theorem 12.1, p. 195]) that U belongs to the (unnormed) linear space $C_{2,\beta}(\Omega)$.

Similar arguments provide regularity "up to the boundary" if $\partial\Omega$ (and its subdivision into Γ_0, Γ_1) and the inhomogenity s in (1) satisfy additional regularity conditions.

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